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Exact Solutions of the Nonlinear Boltzmann Equation

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A review is given of research activities since 1976 on the nonlinear Boltzmann equation and related equations of Boltzmann type, in which several rediscoveries have been made and several conjectures have been disproved. Subjects are (i) the BKW solution of the Boltzmann equation for Maxwell molecules, first discovered by Krupp in 1967, and the Krook–Wu conjecture concerning the universal significance of the BKW solution for the large $(\mathbf{v}, t)$ behavior of the velocity distribution function $f(\mathbf{v}, t)$; (ii) moment equations and polynomial expansions of $f(\mathbf{v}, t)$; (iii) model Boltzmann equation for a spatially uniform system of very hard particles, that can be solved in closed form for general initial conditions; (iv) for Maxwell and non-Maxwell-type molecules there exist solutions of the nonlinear Boltzmann equation with algebraic decrease at $v \to \infty$; connections with nonuniqueness and violation of conservation laws; (v) conjectured super-H-theorem and the BKW solution; (vi) exactly soluble one-dimensional Boltzmann equation with spatial dependence.

KEY WORDS: Boltzmann equation; Maxwell molecules; very hard particles; Krook–Wu conjecture; super-H-theorem.

1. INTRODUCTION

"The Boltzmann equation is Fourier transformed with respect to velocity and the structure of the resulting equation investigated (. . .). A single non-equilibrium solution is constructed in closed analytical form." Thus reads a quotation from the abstract of Krupp's dissertation[(1)] of 1967,[2] in which he derives the following solution of the nonlinear Boltzmann equa-

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[2] Reference due to C. Cercignani.

tion in a spatially uniform gas of Maxwell molecules:

$$f(v,t) = (2\pi s)^{-d/2}\exp(-v^2/2s)\left[1 - \frac{1-s}{2s}\left(d - \frac{v^2}{s}\right)\right] \tag{1a}$$

with

$$s(t) = 1 - \exp\left[-\lambda(t+t_0)\right] \tag{1b}$$

Here λ is a positive quantity, given in terms of an integral of the differential scattering cross section, and d is the number of translational degrees of freedom ($d = 3$ in Krupp's case); t_0 is an arbitrary constant, chosen such that the velocity distribution function $f(v,t) \geqslant 0$ for all times $t \geqslant 0$.

This solution, nowadays called the BKW solution, was rediscovered in 1976 by Bobylev,[2] Krook and Wu,[3] and has created an enormous revival of interest among physicists in exact solution of the Boltzmann equation (more than one hundred publications over the past six years, for which I refer to some recent reviews[4–6]). This activity is gradually fading away. It is my task to review these activities here and highlight the most important developments.

Since the main emphasis will be on spatially uniform systems, I write Boltzmann's famous nonlinear integrodifferential equation in the form

$$\partial_t f(\mathbf{v},t) = \int d\mathbf{w} \int d\hat{\mathbf{n}}\; gI(g,\chi)\left[f(\mathbf{v}',t)f(\mathbf{w}',t) - f(\mathbf{v},t)f(\mathbf{w},t)\right] \tag{2}$$

Here $\mathbf{g} = \mathbf{v} - \mathbf{w}$ is the relative velocity, $I(g,\chi)$ is the differential scattering cross section; $\hat{\mathbf{n}} \parallel \mathbf{g}'$ is a unit vector in the direction of scattering, where $\chi = \arccos(\hat{\mathbf{n}} \cdot \hat{\mathbf{g}})$ is the scattering angle and $\mathbf{v}', \mathbf{w}'$ are postcollisional velocities.

This equation possesses conservation laws for the total number of particles, momentum, and energy, which read in appropriate units:

$$\int d\mathbf{v}\; f(\mathbf{v},t)(1,\mathbf{v},v^2) = (1,0,d) \tag{3}$$

and its solution approaches for long times the Maxwellian velocity distribution

$$f_0(v) = (2\pi)^{-d/2}\exp(-v^2/2) \tag{4}$$

The basic goal of recent research interests is to solve the spatially uniform Boltzmann equation (2) in the case of general intermolecular potentials for a given initial distribution $f(\mathbf{v},0)$ (Cauchy problem). Existence and uniqueness proofs of different types of solutions for different types of interactions have been reviewed recently[7,8] and the mathematical theory of the spatially uniform nonlinear Boltzmann equation is nearly complete. What is lacking are explicit solutions, which can give some understanding of the detailed approach to equilibrium, in particular of the high-energy tail of the distribution function.

To tackle such problems one introduces simplifications of a mathematical and physical nature. A major simplification occurs in the case of Maxwell molecules [interaction potential $U(r) = a/r^4$], where the collision rate $gI(g,\chi) = \alpha(\cos\chi)$ only depends on the scattering angle.[9] However, many more models have been considered recently[4,6] which introduce further simplifications of the quadratic collision term. They are essentially scalar Boltzmann-type equations, which occur in many fields inside and outside physics,[4,10,11] as formulations of stochastic processes in which binary interactions determine the time rate of change of the distribution function, $F(x,t)$. This scalar Boltzmann equation has the general form:

$$\partial_t F(x,t) = \int_x^\infty du \int_0^u dy \left[K(xy;u) F(y,t) F(u-y,t) \right.$$
$$\left. - K(yx;u) F(x,t) F(u-x,t) \right] \tag{5}$$

where $K(xy;u)$ represents the transition rate in binary interactions $(y, u - y) \to (x, u - x)$ conserving the sum of the state variables of the interacting particles. In the kinetic theory of gases the state variable x is the translational energy.

In these equations the total number of particles, $\int F(x,t)\,dx$, and the total energy, $\int xF(x,t)\,dx$, are conserved. The stationary solution can be calculated from the detailed balance condition, obtained by putting $\{\ldots\}$ on the right-hand side of (5) equal to zero.

Next, I will mention a few of the fashionable model-Boltzmann equations, and it should be no surprise that several have been examined before in other fields of science.

First, consider the model, introduced by Tjon and Wu,[12] i.e.,

$$K(xy;u) = 1/u \tag{6a}$$

and more generally the diffuse scattering models, introduced by Ernst[13] and Futcher *et al.*,[14] i.e.,

$$K(xy,u) = \left[x(u-x) \right]^{p-1} / \left[B(p,p) u^{2p-1} \right] \tag{6b}$$

where $B(p,p) = \Gamma^2(p)/\Gamma(2p)$. In probability theory Nishimura[10] had already extensively studied these models (6a), (6b) in 1974 by considering their sequentially soluble moment equations. He further showed that a one-dimensional model Boltzmann equation, introduced by Kac,[15] can be cast into the form of a scalar Boltzmann equation with a transition kernel (6b) with $p = 1/2$.

Another Maxwell-type model of Ref. 16, in which the directions of the relative velocity before and after collision are at right angles, corresponds to the kernel

$$K(xy;u) = \delta(x - \tfrac{1}{2}u) \tag{6c}$$

The same model, together with its moment equation, has already been studied in 1963 by Curl[11] for the time evolution of the drop size distribution in the theory of disperse phase mixing.[3]

The previous examples (6a)–(6c) have in common that the loss term on the right-hand side of (5) can be reduced to $-\omega F(x,t)$, where the total collision frequency ω is a constant on account of particle conservation. Therefore, these models are all Maxwell-type models, to which I return in Sections 2 and 3.

As a last example I mention the very hard particle (VHP) model, introduced by Rouse and Simons[16] in its discrete form and by Hendriks and Ernst[17] in its continuous form. It will be considered in Section 4, and corresponds to the transition kernel

$$K(xy;u)=1 \tag{7}$$

Here the loss term in (5) reduces to $-(x+1)F(x,t)$ on account of particle and energy conservation, and the total collision frequency, $\omega(x)=x+1$, depends on energy. Hence the VHP model is not a Maxwell model.

The subsequent sections are organized as follows: the BKW solution and the Krook–Wu conjecture are discussed in Section 2. In Section 3 I discuss moment equations and polynomial expansions for Maxwell models. The Cauchy problem for the VHP model is solved in closed form in Section 4. Power law solutions, violation of conservation laws and nonuniqueness are discussed in Section 5, and Section 6 deals with the super-H-theorem. A soluble model-Boltzmann equation for a spatially nonuniform system is treated in Section 7.

2. BKW SOLUTION AND KROOK–WU CONJECTURE

Fourier transformation of the Boltzmann equation for Maxwell molecules yields an elegant method to obtain the BKW solution. Bobylev[2] has shown that the characteristic function or Fourier transform of the distribution function:

$$\phi(\mathbf{k},t)=\int d\mathbf{v}\, f(\mathbf{v},t)\exp(-i\mathbf{k}\cdot\mathbf{v}) \tag{8}$$

satisfies the following equation:

$$\partial_t\phi(\mathbf{k},t)=\int d\hat{\mathbf{n}}\,\alpha(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})\left\{\phi\left(\tfrac{1}{2}k(\hat{\mathbf{k}}+\hat{\mathbf{n}}),t\right)\phi\left(\tfrac{1}{2}k(\hat{\mathbf{k}}-\hat{\mathbf{n}}),t\right)-\phi(\mathbf{k},t)\phi(\mathbf{0},t)\right\} \tag{9}$$

This equation represents a drastic simplification of the Boltzmann equation for Maxwell molecules, since the collision term in (2) is a $(2d-1)$-fold integral, which is reduced to a $(d-1)$-fold integral in the Fourier represen-

[3] Reference due to R. M. Ziff.

tation (9). For an isotropic distribution function, $f(|\mathbf{v}|, t)$, the characteristic function $\phi(k, t)$ depends only on the length $|\mathbf{k}| = k$.

Bobylev showed the following symmetry properties of this equation:

(i) If $\phi(\mathbf{k}, t)$ is a solution of (9), then $\exp(-\frac{1}{2}sk^2)\phi(\mathbf{k}, t)$ is also a solution of the same equation (*Bobylev symmetry*).

(ii) Equation (9) is invariant under a group of similarity transformations. Consequently, it allows *similarity solutions* of the form $\phi(\mathbf{k}, t) = \Phi(\mathbf{k}e^{-\lambda t/2})$, in which the number of independent variables is reduced by one.

Now it is easy to find the similarity solution $\phi(\mathbf{k}) = (1 + ak^2)\exp(bk^2)$, which becomes (after imposing the physically relevant boundary conditions[6]):

$$\phi(\mathbf{k}, t) = \left(1 - \tfrac{1}{2}\beta k^2\right)\exp\left[-\tfrac{1}{2}k^2(1 - \beta)\right] \tag{10a}$$

with

$$\beta(t) = \exp\left[-\lambda(t + t_0)\right] \tag{10b}$$

and

$$\lambda = \tfrac{1}{4}\int d\hat{\mathbf{n}}\, \alpha(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}})\left[1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{k}})^2\right] \tag{10c}$$

This similarity solution is a special case of the canonical expansion, used by Krupp[1] for general interaction potentials:

$$\phi(k, t) = \exp\left(-\tfrac{1}{2}k^2\right)\prod_{m=0}^{\infty}\left[1 - \alpha_m(t)k^2\right]\exp\left[\alpha_m(t)k^2\right] \tag{11}$$

In (10) only $\alpha_0(t)$ is nonvanishing. Inversion of the Fourier transform (10) directly yields the BKW solution (1).

For more general Maxwell models one has found by a variety of methods a special BKW solution of the general form[6]

$$F(x, t)/F_0(x) = \{A + xB\}e^{-\alpha x} \tag{12}$$

where $F_0(x)$ is the stationary solution of the scalar Boltzmann equation (5), and A, B, and α are relatively simple functions of time.

Such exact solutions are of interest on several scores: rareness, formation of Maxwell tail, and Krook–Wu conjecture. I start with a discussion of the second point.

In Fig. 1 the BKW solution $R(v, t) = f(v, t)/f_0(v)$ is plotted as a function of the velocity for several values of the scaled time $\tau = \lambda(t + t_0)$. Notice the extremely slow (nonuniform) approach to the equilibrium solution $R(v, \infty) = 1$ in the high-energy tail of the distribution function. For thermal velocities, i.e., $v \lesssim 1$, the typical relaxation time is, of course, the mean free time, proportional to $1/\lambda$. Typical relaxation times, $t_0(v)$, at large

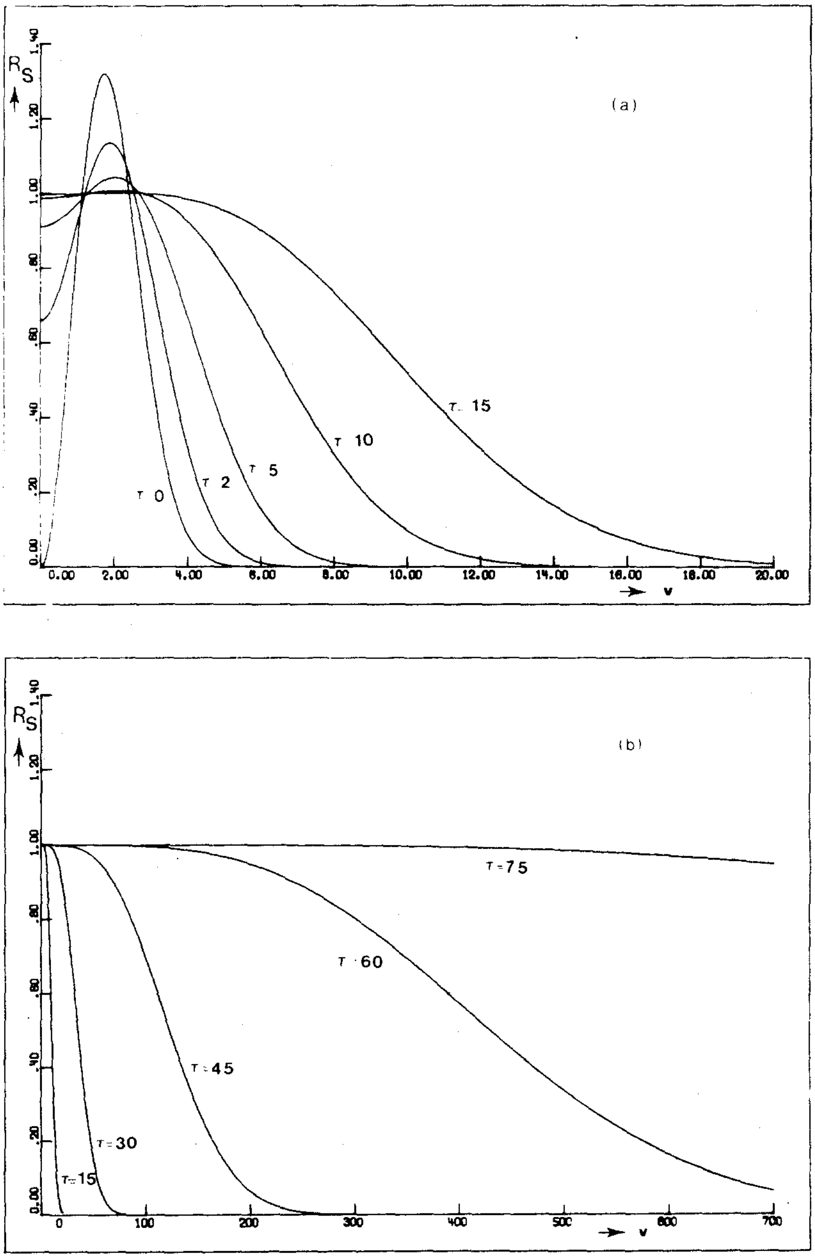

Fig. 1. The ratio $R(v, t) = f(v, t)/f_0(v)$ for the BKW mode (1) of Maxwell molecules as a function of velocity with $\tau = \lambda(t + t_0)$.[3]

velocities can be obtained from (10) by taking v large and $\beta(t)$ small, i.e.,

$$R(v,t) = f(v,t)/f_0(v) \simeq \left(1 + \tfrac{1}{2}\beta v^2\right)\exp\left(-\tfrac{1}{2}\beta v^2\right) \tag{13}$$

An estimate of the characteristic time $t_0(v)$ follows from the relation $v^2\beta(t_0) \simeq 1$, yielding $t_0(v) \simeq (\log v)/\lambda$. Equation (13) shows that *for a given large time t one can always find a sufficiently large velocity*, where the factor $\exp(-\frac{1}{2}\beta v^2)$ is not close to unity, and *where the solution* (13) *cannot be linearized around its equilibrium value.*

This nonuniform approach to the Maxwellian shows the inadequacy of the linearized Boltzmann equation for describing relaxation phenomena in the high-energy tail of the distribution function. Similar nonuniform relaxation processes have been found in the exactly solvable very hard particle model of Section 4.

Next, I turn to the Krook–Wu conjecture.[3] The possible importance of the BKW solution has been considerably increased by the following conjecture formulated by Krook and Wu: "An arbitrary initial state tends first to relax towards a state characterized by the similarity solution. The subsequent stage of the relaxation is essentially represented by the similarity solution with an appropriate phase."

The first evidence against the validity of this conjecture was found by Tjon[18] from the scalar Boltzmann equation (5) with transition kernel (6a). He obtained numerical solutions for initial states that relax to equilibrium in a manner qualitatively different from the BKW solution. His results can be seen in Fig. 2, where the solid lines show an interesting relaxation phenomenon, not present in the BKW solution. The high-energy tail—initially empty—builds up at a fast rate, overshoots its final equilibrium value, and exhibits a transient high-energy tail in the distribution function with an overpopulation, significantly larger than the Maxwell value. However, one should keep in mind that the total fraction of particles in the energy range under discussion may easily be as low as 10^{-50}. The dotted lines in Fig. 2 represent the decay of an initial distribution, exhibiting a monotonic increase of the distribution function at large values of the energy, as in the BKW solution.

Analytical evidence against the validity of the conjecture was first given by Alexanian[19] and Hauge,[20] who formulated conditions on the initial distribution $f(v,0)$, determining whether the final approach to the Maxwellian would be from above or from below. For a review I refer to Refs. 4 and 6.

On the basis of the numerical and analytic evidence against the conjecture and in accordance with most publications on the subject,[4,6,19–21] I conclude that the conjecture is incorrect with very high

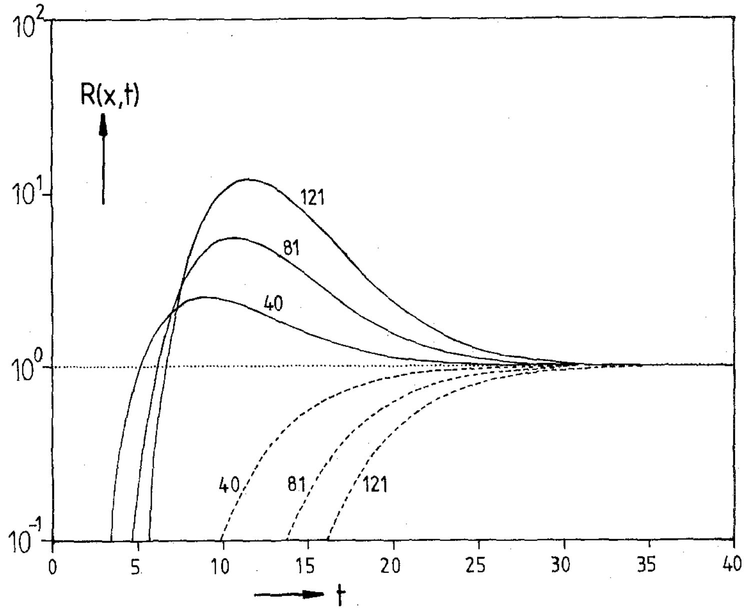

Fig. 2. The ratio $R(x, t) = f(v, t)/f_0(v)$ for the two-dimensional Maxwell model (6a) as a function of the time t at various values of the energy $x = \frac{1}{2}v^2$.[18] The initial distributions consist of two delta peaks, located at $x = 0.4$ and 2.0 (dashed lines) and at $x = 0.4$ and 3.6 (solid lines).

probability. Its main accomplishment was the stimulus it provided in searching for exact solutions of the nonlinear Boltzmann equation.

In recent years more careful estimates for the asymptotic behavior of $f(v, t)$ at large v and t have been given by Bobylev[22] for special classes of initial distributions.

The transient overpopulation phenomenon, seen by Tjon,[18] has also been studied in a dense system of Lennard-Jones particles by means of molecular dynamics,[23] and the results are in qualitative agreement with the criteria of Alexanian and Hauge.

3. MOMENT EQUATIONS AND POLYNOMIAL EXPANSIONS

This section, like the previous one, deals exclusively with Maxwell-type models. In this case the general solution of the nonlinear Boltzmann equation will be given in the form of an expansion in orthogonal polynomials. The coefficients in this expansion, called polynomial moments, as well as the ordinary moments, can be found sequentially, given their initial values, i.e., $f(\mathbf{v}, 0)$. Therefore, one possesses the general solution of the initial-value problem within a certain Hilbert space, provided the series converges.

In hindsight one can, of course, say that the essential ideas were already present in the older literature. In 1955 Kac[15] proposed a one-dimensional model-Boltzmann equation of Maxwell type, and showed that the general solution can be given as series in the Hermite polynomials. The coefficients in this series, the Hermite moments, satisfy a recursively soluble set of differential equations.

As early as 1949 Grad[24] showed that the first few (tensor) Hermite moments satisfy a similar set of equations, and Truesdell[25] found in 1956 the same for the ordinary velocity moments. Similar recursive systems of moment equations for the kernels in (6c) and (6b) have been discussed, respectively, in 1966 by Curl,[11] and in 1974 by Nishimura.[10]

In recent years ordinary and polynomial moment equations for Maxwell models have been extensively discussed in the literature.[14,20,21,26,27] The only essentially new result of recent years seems to be the observation that the ordinary moments and polynomial moments (both properly normalized) satisfy exactly the same set of moment equations.[13,6]

In order to discuss these results I have to restrict the allowed distribution functions $f(\mathbf{v}, t) = R(v, t) f_0(v)$ to the Hilbert space with norm

$$N(t) = \|R\|^2 = \int d\mathbf{v}\, f_0(v) |R(\mathbf{v}, t)|^2 < \infty \tag{14}$$

which is sufficient for the existence of all moments. Power law solutions, for which only a finite number of moments exists, will be examined in Section 5.

For convenience I only consider isotropic velocity distributions, and present a short derivation of the polynomial expansion for that case. The characteristic function depends now only on the length of $\mathbf{k}$, and generates the *normalized moments* $M_n(t)$ through the Taylor expansion:

$$\phi(k, t) = \sum_{x=0}^{\infty} \left(-\tfrac{1}{2}k^2\right)^n M_n(t)/n! \tag{15}$$

with

$$M_n(t) = \left[\Gamma\left(\tfrac{1}{2}d\right)/\Gamma\left(\tfrac{1}{2}d + n\right)\right] \int d\mathbf{v} \left(\tfrac{1}{2}v^2\right)^n f(\mathbf{v}, t) \tag{16}$$

The conservation laws guarantee the relations $M_0(t) = M_1(t) = 1$. Inserting (15) into (9) and equating the coefficients of equal powers of k on both sides of the equation yields the moment equation for Maxwell molecules:

$$\dot{M}_n + \Lambda_n M_n = \sum_{k=1}^{n-1} \mu_{nk} M_k M_{n-k} \tag{17}$$

where

$$\mu_{n+l,l} = \frac{(n+l)!}{n!\,l!} \int d\mathbf{n}\, \alpha(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) \left(\frac{1 + \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}}{2} \right)^l \left(\frac{1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}}{2} \right)^n \tag{18}$$

$$\Lambda_n = \mu_{00}(1 + \delta_{n0}) - \mu_{n0} - \mu_{nn} \tag{19}$$

The most important feature of these nonlinear moment equations is of course that each $M_n(t)$ can in principle be found sequentially, where $M_n(0)$ is given through (16) and $f(v, 0)$. This property of Maxwell molecules was already known for a long time; perhaps even to Maxwell himself.[25]

As the next step I write an alternative expansion of $\phi(k, t)$, viz.,

$$\phi(k, t) = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}k^2\right)^n e^{-k^2/2} c_n(t)/n! \tag{20}$$

where $c_0(t) = 1$ and $c_1(t) = 0$ by virtue of the conservation laws, and $c_n(\infty) = \delta_{n0}$. As a consequence of Bobylev's symmetry property discussed below (9), *the coefficients $c_n(t)$ satisfy exactly the same set of equations* (17) *as the ordinary moments $M_n(t)$*. The expansion (20) can be Fourier-inverted term by term and yields the Laguerre series for the distribution function for general dimensionality $d = 2m$:

$$f(v, t) = f_0(v) \left\{ 1 + \sum_{n=2}^{\infty} c_n(t) L_n^{(m-1)}\left(\tfrac{1}{2}v^2\right) \right\} \tag{21}$$

The coefficients $c_n(t)$ are the polynomial moments

$$c_n(t) = \frac{n!\,\Gamma(m)}{\Gamma(m+n)} \int d\mathbf{v}\, L_n^{(m-1)}\left(\tfrac{1}{2}v^2\right) f(v, t) \tag{22}$$

which can be found recursively from (17), given $c_n(0)$. Thus we have the general solution of the Cauchy problem in the Hilbert space (14).

For the scalar Boltzmann equations of Maxwell type one has found similar expansions in orthogonal polynomials, and similar recursively soluble moment equations. Extensions to anisotropic velocity distributions for Maxwell molecules have been given in Refs. 28 and 29.

The above solutions have been extensively studied numerically,[20,21,26] and convergence proofs of these expansions have been given for special classes of initial distributions.[21,29,30]

4. VERY HARD PARTICLE (VHP) MODEL

This model describes a gas of particles with only two translational degrees of freedom. The scattering laws are stochastic (i.e., total momentum is not conserved), such that binary collisions $(\mathbf{v}, \mathbf{w} \rightarrow \mathbf{v}', \mathbf{w}')$ occur with a probability $\delta(E - E')$, where $E = \frac{1}{2}v^2 + \frac{1}{2}w^2$.[16,17] Alternatively,[17] the model may be interpreted as deterministic (total energy *and* total momentum are conserved during two-particle scattering), and the differential scattering cross section is given by $I(g,\chi) = \frac{1}{8}g|\sin\chi|$.

The interest of this model lies in its solubility. It is the only model for which the energy-dependent distribution function of the nonlinear Boltzmann equation has been obtained in closed form for arbitrary initial conditions. Unfortunately, the model has the unphysical feature that the scattering cross section increases as $(\text{energy})^{1/2}$, whereas in real systems it is bounded by a constant, as in hard-sphere systems. Hence the relaxation at high energies will be too fast.

It is most convenient to discuss the model in terms of the energy distribution function $F(x,t)$. It satisfies a scalar Boltzmann equation (5) with transition kernel (7):

$$(\partial_t + x + 1)F(x,t) = \int_x^\infty du \int_0^u dy\, F(y,t)F(u-y,t) \tag{23}$$

where the loss term has been simplified by the help of particle and energy conservation. The convolution structure of the collision term suggests the use of Laplace transformations. Therefore, I introduce

$$G(z,t) = \int_0^\infty dx\, e^{-zx}F(x,t) \tag{24}$$

and transform the Boltzmann equation (23) into

$$(\partial_t - \partial_z + 1)G = \frac{1}{z}(1 - G^2) \tag{25}$$

This is a first-order nonlinear partial differential equation, which is soluble by standard methods. Its general solution reads:

$$G(z,t) = \frac{\phi(z+t) + (z-1)e^{-t}}{(z+1)\phi(z+t) - e^{-t}} \tag{26}$$

where the arbitrary function $\phi(z)$ can be determined through the initial condition, $G(z,0)$. Thus, the energy distribution function,

$$F(x,t) = (2\pi i)^{-1} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dz\, e^{zx}G(z,t) \tag{27}$$

constitutes the solution to the Cauchy problem for the VHP model. This solution is unique provided the admitted functions $F(x,t)$ are restricted to functions decreasing faster than x^{-3} at large x (see Section 5).

This closed form solution has been analyzed in great detail and many questions regarding approach to equilibrium, formation of Maxwell tails, etc. have been answered.[17,6]

For comparison I also quote the scalar Boltzmann equation for the related Maxwell model (6a):

$$(\partial_t + 1)F(x,t) = \int_x^\infty \frac{du}{u} \int_0^u dy\, F(y,t)F(u-y,t) \tag{28}$$

which becomes after Laplace transformation[12]

$$(\partial_t + 1)\partial_z(zG) = G^2 \tag{29}$$

This is a second-order nonlinear partial differential equation, of which the general solution is not known in closed form. Of course, there exists again the Laguerre series expansion of the solution, which forms the subject of many papers.[14,21,26)]

5. POWER LAW SOLUTIONS FOR MAXWELL AND NON-MAXWELL MODELS

In Section 3 I considered solutions of the Boltzmann equation for Maxwell models, which belong the the Hilbert space with norm (14). However, this integral (14) has no physical meaning. The only physical requirements are that the total number of particles and total energy are finite.

Therefore, Bobylev[2] has considered solutions (eigenfunctions) of the linearized Boltzmann equation for Maxwell molecules with a power law decrease at large velocities. Cornille and Gervois[31] have constructed similar solutions to the nonlinear Boltzmann equation for a Maxwell-type model. The above solutions do not belong to the Hilbert space,[14] the ordinary and polynomial moments do not exist beyond a certain order, and the Laguerre series (21) no longer represents the solution of the nonlinear Boltzmann equation for Maxwell models.

First I consider *Maxwell models* and show that solutions, such as found in Refs. 2 and 31, can be constructed for general dimensional Maxwell models, provided the total number of particles and total energy are finite. For convenience I take isotropic velocity distributions, so that $\phi(k,t)$ in (8) depends only on $|\mathbf{k}|$. Now, I look for solutions of (9) in the form

$$\phi(k,t) = e^{-k^2/2}\left\{1 + \sum_{a>1} C_a(t)k^{2a}\right\} \tag{30}$$

where $a > 1$ since the total energy must exist. Insertion of (30) into (9) leads to the relations

$$\sum_{a>1} (\dot{C}_a + \Lambda_a C_a) k^{2a}$$

$$= \sum_{a,a'>1} \left(\tfrac{1}{2}k^2\right)^{a+a'} C_a C_{a'} \int d\hat{\mathbf{n}}\, \alpha(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})(1 + \hat{k} \cdot \hat{n})^a (1 - \hat{k} \cdot \hat{n})^{a'} \quad (31)$$

with Λ_a defined in (19). There exist several choices of the index a, such that (31) reduces to a recursively soluble set of equations.[31] A possible choice is

$$a = a(n) = (n + M)/\xi \qquad (n = 0, 1, 2 \ldots) \qquad (32)$$

where M is a positive integer and $\xi > M$. After having determined $C_{a(n)}(t)$, Fourier inversion of (30) yields

$$f(v,t) = f_0(v) \left\{ 1 + \sum_{n=0}^{\infty} b_n(t)\, {}_1F_1\left(-a(n); m; \tfrac{1}{2}v^2\right) \right\} \qquad (33)$$

with

$$b_n(t) = C_{a(n)}(t)\Gamma(m + a(n))/\Gamma(m) \qquad (34)$$

where ${}_1F_1$ is a confluent hypergeometric function and $m = d/2$. If $a = n$ the function ${}_1F_1$ reduces to the Laguerre polynomial $L_n^{(m-1)}$, and (33) reduces to (21), which is contained in the Hilbert space (14). For general $a \neq$ integer, the functions $f_0(v)\, {}_1F_1(\frac{1}{2}v^2)$ have power law decrease like v^{-2a-2} at $v \to \infty$. The corresponding eigenvalues Λ_a belong to the continuous part of the spectrum, which is continuous down to zero (corresponding to conserved quantities). Consequently, in the class of initial conditions with power law decrease at $v \to \infty$, the asymptotic relaxation rate (related to the spectrum of the linearized Boltzmann equation) may be arbitrarily slow.

The previous discussion shows that in the case of Maxwell models there exist solutions of the Boltzmann equation, having powerlike decrease at large v and lying outside the usual Hilbert space (14).

For *non-Maxwell models* the situation is different. Cornille and Gervois[32] have shown the existence of solutions (eigenfunctions) of the linearized Boltzmann equation in the case of hard spheres, decaying like v^{-6} as $v \to \infty$. However, such solutions must be rejected because they violate energy conservation. It was further shown[33–34] that the Cauchy problem for the Boltzmann equation does not have a unique solution if the above power law solutions are admitted. Such solutions correspond in fact to an influx of particles from infinite energies at an arbitrary rate.[34]

To summarize the present situation on power law solutions of the Boltzmann equations I list some properties (only part of which have been proven[32]). Consider an interparticle potential with $gI(g,\chi) = g^\gamma \alpha(\chi)$ (where Maxwell molecules have $\gamma = 0$, hard spheres $\gamma = 1$, and very hard particles $\gamma = 2$); then

i. For particles softer than Maxwell molecules ($\gamma \leqslant 0$), the solution of the initial value is unique and energy conservation holds provided admitted solutions $f(v,t)$ decrease faster than $1/v^{d+2}$ at $v \to \infty$.

ii. For hard interparticle interactions ($\gamma > 0$), uniqueness and conservation laws are guaranteed provided admitted solutions decrease faster than $1/v^{d+2+\gamma}$.

iii. When solutions decaying like $1/v^{d+2+\gamma}$ are admitted for $\gamma > 0$, there exist an infinite number of solutions to the initial value problem.

iv. There exists a continuous spectrum of the linearized Boltzmann equation such that the zero eigenvalue is not an isolated part of the spectrum. For $\gamma > 0$, the corresponding eigenfunctions, decaying for large v like $1/v^{d+2+\gamma}$, violate energy conservation. Furthermore these eigenfunctions do not belong to the standard Hilbert space (14).

6. SUPER-H-THEOREM

According to Boltzmann's H-theorem the approach to equilibrium of any solution $f(\mathbf{v},t)$ of the Boltzmann equation (2) is accompanied by a monotonic decrease of the H-function:

$$H(t) = \int d\mathbf{v}\, f(\mathbf{v},t) \ln f(\mathbf{v},t) \tag{35}$$

Thus $dH/dt \leqslant 0$, the equality holding for equilibrium. For the scalar Boltzmann equation (5) the H-function is defined as[10]

$$H(t) = \int dx\, F(x,t) \ln\big(F(x,t)/F_0(x)\big) \tag{36}$$

where $F_0(x)$ is a stationary solution of (5).

Some years ago a number of extensions of the standard H-theorem were proposed by McKean[35] and Harris.[36] They speculated on the possibility that $H(t)$ might be completely monotonic, i.e.,

$$(-)^n d^nH/dt^n \geqslant 0 \tag{37}$$

for all $n = 1, 2, \ldots$. This super-H-theorem, as McKean conjectured, might single out the H-function as the correct nonequilibrium entropy from a large class of functions that also lead monotonically to the correct value of the equilibrium entropy.

Since all proofs and disproofs of the super-H-theorem, given before, apply only to extremely simplified model-Boltzmann equations, Ziff, Merajver, and Stell[37] investigated the alternating derivative property for the BKW solution (1), which is an exact solution to a nontrivial model-

Boltzmann equation. These authors transformed dH/dt for the BKW solution into the form

$$\frac{dH}{d\tau} = -\frac{1}{\Gamma(d/2)(e^{\tau}-1)^2}\int_0^{\infty}\frac{x^{1+d/2}e^{-x}}{(x+y)^2}\,dx \tag{38}$$

where $\tau = \lambda(t + t_0)$, $x = v^2/2$, and $y = e^{\tau} - 1 - d/2$, and showed that (37) is valid up to 30 with d-values ranging from 1 to 6. They concluded that their results strongly support the validity of the super-H-theorem.

However, shortly thereafter Olaussen,[(38)] Garret[(39)] and Lieb[(40)] disproved the theorem. It is amusing to mention Olaussen's estimate that (37) ceases to be valid for n-values around one hundred.

7. A SOLUBLE MODEL BOLTZMANN EQUATION WITH SPATIAL DEPENDENCE

So far all models considered were restricted to spatially uniform systems. For the spatially nonuniform case the only soluble model Boltzmann equation has been constructed recently by Ruijgrok and Wu.[(41)] They consider particles on a line with positions $x \in (-\infty, \infty)$ and velocities $(+1, -1)$, where $f_+(x,t)$ and $f_-(x,t)$ are the distribution functions for these velocities. The Boltzmann equation for their model has the form

$$\begin{aligned}(\partial_t + \partial_x)f_+ &= f_+ f_- - \alpha f_+ + \beta f_- \\ (\partial_t - \partial_x)f_- &= -f_+ f_- + \alpha f_+ - \beta f_-\end{aligned} \tag{39}$$

where α and β are two positive constants. There are three collision processes: $(+\,-)\to(+\,+)$: $(+)\to(-)$ and $(-)\to(+)$ with rate constants 1, α, and β, respectively.

The absence of the restituting collisions $(+\,+)\to(+\,-)$ means violation of detailed balance and of microscopic time reversal invariance. This fact is the basic reason for solubility of the present model.

For the explicit form of the solution for general initial conditions I refer to the original literature. If $\alpha < \beta$ there exists a unique spatially uniform equilibrium solution; if $\alpha > \beta$, then (39) admits special solutions that are shock-waves. As applications of their general solution the authors study the asymptotic behavior of $f_{\pm}(x,t)$ for large t, and the interaction of shock waves.

To summarize, I list as conclusions on exact solutions of the *spatially uniform* nonlinear Boltzmann equation the following:

- The special BKW-solution and the general Laguerre series solution for Maxwell molecules and similar models are essentially rediscoveries of known results.

- The Krook–Wu conjecture on the universal significance of the BKW solution and the McKean–Harris conjecture on the super-H-theorem do not hold.
- Laguerre series solutions for Maxwell molecules are adequate at thermal velocities, but inadequate at very large velocities. It would, therefore, be desirable to have closed form solutions or asymptotic solutions ($v \to \infty$) for Maxwell molecules and other more realistic interaction models.
- The Cauchy problem for the very hard particle model has been solved in closed form for general initial conditions, but the interactions are physically unrealistic.
- For the nonlinear Boltzmann equation in the *spatially nonuniform* case there exists an exactly soluble model of Ruijgrok and Wu.

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